

On the Number of Representations of Integers by various Quadratic and Higher Forms

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Abstract

We give formulas for the number of representations of non negative integers by various quadratic forms. We also give evaluations in the case of sum of two cubes (cubic case) and the quintic case, as well. We introduce a class of generalized triangular numbers and give several evaluations. Finally, we present a mean value asymptotic formula for the number of representations of an integer as sum of two squares known as the Gauss circle problem.

keywords: Quadratic Forms; Diophantine Equations; Sums of Squares; Asymptotics; Cubic Form; Quintic Form; Special Functions

1 Introduction.

The study of quadratic forms has been built up by many great mathematicians such as Euler, Gauss, Dirichlet, Liouville, Eisenstein, Glaisher, Ramanujan among others. This theory has applications to a wide number of areas in modern mathematics including Gauss' circle problem in higher dimensions, class number theory, algebraic geometry, elliptic and theta functions, the Fermat-Wiles theorem, Eisenstein series and many other (see [2-10]).

In this article using simple arguments we try to address the problem.

We start with $K(x)$, the complete elliptic integral of the first kind, given by

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2(\theta)}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \quad (1)$$

where ${}_2F_1$ is Gauss hypergeometric function.

In terms of Weber's $\lambda(\tau)$ -modular function (see [3],[4])

$$\lambda(\tau) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8, \quad (2)$$

where $q = e^{i\pi\tau}$, $Im(\tau) > 0$, $\tau = \sqrt{-r}$, the singular modulus $k = k_r$, $r > 0$ is

$$k_r^2 = \lambda(\tau) = \left(\frac{\theta_2(q)}{\theta_3(q)} \right)^4, \quad (3)$$

with

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \text{ and } \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, |q| < 1 \quad (4)$$

Also $k = k_r$, $0 < k < 1$ is the solution of the equation

$$\frac{K(\sqrt{1-k_r^2})}{K(k_r)} = \sqrt{r} \quad (5)$$

As usual we set $K = K(k_r)$ (the complete elliptic integral at singular values) and $K' = K(k'_r)$, where $k'_r = \sqrt{1-k_r^2}$ is the complementary singular modulus. The Fourier expansion for the Jacobi elliptic function dn (see [3] p.51-53) is

$$\text{dn}(q, u) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos(2nz) \quad (6)$$

where $z = \left(\frac{\pi}{2K}\right) u$ lies in the strip $|Im(z)| < \frac{\pi}{2} Im(\tau)$, $\tau = i\frac{K'}{K}$.

A very interesting connection between number theory and the theory of elliptic functions stems from Jacobi's famous theorem

Theorem 1. (Jacobi [3])

If $q = e^{-\pi\sqrt{r}}$, $r > 0$, then

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} \quad (7)$$

This theorem plays a key role in the theory of elliptic functions and we shall use it here in our investigation of quadratic forms of general type. It is very easy to see by setting $u = 0$ in (6), using $\text{dn}(q, 0) = 1$ and then multiplying both sides of (6) by $2K/\pi$, that

$$\begin{aligned} \frac{2K}{\pi} &= 1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}} = 1 + 4 \sum_{m=1}^{\infty} q^m \sum_{l=0}^{\infty} (-1)^l q^{2ml} = \\ &= 1 + 4 \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l q^{(2l+1)m}. \end{aligned}$$

Writing $n = (2l+1)m$, $d = 2l+1$, so $l = (d-1)/2$, if d runs through the odd divisors of n we have

$$\frac{2K}{\pi} = 1 + 4 \sum_{n=1}^{\infty} \left[\sum_{d-\text{odd}, d|n} (-1)^{\frac{d-1}{2}} \right] q^n \quad (8)$$

We define $\delta_0(n) = 1$, if $n = 0$ and $\delta_0(n) = 4 \sum_{d-\text{odd}, d|n} (-1)^{\frac{d-1}{2}}$, if $n \geq 1$. If $r(n)$ denotes the number of representations of n by the form

$$n = x^2 + y^2, (x, y \in \mathbf{Z}),$$

then, if we consider the fact that

$$\theta_3(q)^2 = \sum_{n=-\infty}^{\infty} q^{n^2} \sum_{m=-\infty}^{\infty} q^{m^2} = \sum_{n,m=-\infty}^{\infty} q^{n^2+m^2} = \sum_{n=0}^{\infty} r(n)q^n$$

and apply Jacobi's Theorem 1, we get

Theorem 2. (Jacobi [8])

For $n = 1, 2, \dots$ we have

$$r(n) = 4 \sum_{d-\text{odd}, d|n} (-1)^{\frac{d-1}{2}} \quad (9)$$

and $r(0) = 1$.

2 Generalizations of Jacobi's two-square theorem

Suppose we have two positive integers A, B , with $\gcd(A, B) = 1$, and let $r_{A,B}(n)$ denote the number of representations of n by the quadratic form

$$n = Ax^2 + By^2 \quad (10)$$

Then

$$\theta_3(q^A)^2 \theta_3(q^B)^2 = \left(\sum_{n,m=-\infty}^{\infty} q^{An^2+Bm^2} \right)^2 = \left(\sum_{n=0}^{\infty} r_{A,B}(n)q^n \right)^2$$

But, also

$$\begin{aligned} \theta_3(q^A)^2 \theta_3(q^B)^2 &= \left(\sum_{n=0}^{\infty} r(n)q^{nA} \right) \left(\sum_{m=0}^{\infty} r(m)q^{mB} \right) = \\ &= \sum_{n=0}^{\infty} \left(\sum_{kA+lB=n} r(k)r(l) \right) q^n \end{aligned}$$

The linear Diophantine equation $kA + lB = n$ has solutions for all n since $\gcd(A, B) = 1|n$.

We now introduce the transformation T , which assigns the Taylor coefficient f_n of a function $f(q)$ to the Taylor coefficient $(\sqrt{f})_n$ of its square root $\sqrt{f(q)}$, i.e.

$$(\sqrt{f})_n = T(f_n).$$

The transform T can be evaluated by using Faa Di Bruno's Formula (see [1] p.823), which in this case is

$$T(f_n) = \sum_{m=0}^n h_m(f_0) \sum' \prod_{j=1}^n \frac{f_j^{a_j}}{a_j!} \quad (11)$$

where the prime on the sum means that we sum over all non-negative integers a_j such that $a_1 + 2a_2 + 3a_3 + \dots + na_n = n$ and $a_1 + a_2 + a_3 + \dots + a_n = m$. The function $h_m(x) = (-1)^m x^{1/2-m} \left(\frac{-1}{2}\right)_m$, $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$, $m = 1, 2, \dots$

With the above notation we can proceed to

Proposition 1.

Given two positive integers A, B with $\gcd(A, B) = 1$ the number of the representations of $n = 1, 2, \dots$ by the form $Ax^2 + By^2$ is exactly

$$r_{A,B}(n) = T \left(\sum_{kA+lB=n} r(k)r(l) \right) \quad (12)$$

Note that $r_{A,B}(0)$ is obviously 1.

Proposition 2.

Given two positive integers A, B with $\gcd(A, B) = 1$, the number of representations of $n = 1, 2, \dots$ by the form $Ax^2 + By^2$ is exactly

$$r_{A,B}(n) = \left[\frac{1}{n!} \frac{d^n}{dq^n} \sqrt{\sum_{t=0}^n \left(\sum_{kA+lB=t} r(k)r(l) \right) q^t} \right]_{q=0} \quad (13)$$

In the same way as above we can prove

Theorem 3.

If A_1, A_2, \dots, A_N are positive integers such that $\gcd(A_1, A_2, \dots, A_N) = 1$, the number of the representations of $n = 1, 2, \dots$ by the form $\sum_{k=1}^N A_k x_k^2$ is exactly

$$r_2(N, n) = T \left(\sum_{k_1 A_1 + k_2 A_2 + \dots + k_N A_N = n} r(k_1)r(k_2) \dots r(k_N) \right) \quad (14)$$

and

$$\begin{aligned} r_2(N, n) &= \\ &= \left[\frac{1}{n!} \frac{d^n}{dq^n} \sqrt{\sum_{t=0}^n \left(\sum_{k_1 A_1 + k_2 A_2 + \dots + k_N A_N = t} r(k_1)r(k_2) \dots r(k_N) \right) q^t} \right]_{q=0} \end{aligned} \quad (15)$$

Proposition 3.

Consider the non-homogeneous quadratic form

$$Ax^2 + By^2 + Cx + Dy + E \quad (16)$$

with A, B positive integers, C, D, E general integers, $\gcd(A, B) = 1$, and $C \equiv 0 \pmod{2A}$, $D \equiv 0 \pmod{2B}$. Then n has exactly

$$r_{A,B} \left(n + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right)$$

representations by (16).

Proof.

Write $C = -2L_1A$ and $D = -2L_2B$. Then $n = Ax^2 + By^2 + Cx + Dy + E$ is equivalent to $n = A(x - L_1)^2 + B(y - L_2)^2 - AL_1^2 - BL_2^2 + E$ and the number of representations of n by (16) is equal to the number of representation of $n + AL_1^2 + BL_2^2 - E = n + \frac{C^2}{4A} + \frac{D^2}{4B} - E$, by $Ax^2 + By^2$. qed

Application 1.

Let A, B, C, D be as in Proposition 3, then

$$\sum_{n=-\infty}^{\infty} q^{An^2+Cn} \cdot \sum_{n=-\infty}^{\infty} q^{Bn^2+Dn} = 2\pi^{-1} q^{-n_0} K(k_r) \sqrt{m_{A,r} m_{B,r}}$$

where $n_0 = \frac{C^2}{4A} + \frac{D^2}{4B}$ and $q = e^{-\pi\sqrt{r}}$. The function $m_{n,r} = \frac{K(k_{n^2r})}{K(k_r)}$ is called a multiplier (see [4] pg.136) and takes algebraic values when n is a positive integer and r is rational.

Proof.

From Proposition 3 we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{An^2+Cn} \cdot \sum_{n=-\infty}^{\infty} q^{Bn^2+Dn} &= \sum_{n,m=-\infty}^{\infty} q^{An^2+Bm^2+Cn+Dm} = \\ &= \sum_{n=0}^{\infty} r_{A,B}(n) q^{n-n_0} = q^{-n_0} \sum_{n=0}^{\infty} r_{A,B}(n) q^n = q^{-n_0} \vartheta_3(q^A) \vartheta_3(q^B) = \\ &= q^{-n_0} \sqrt{\frac{2K(k_{A^2r}) 2K(k_{B^2r})}{\pi^2}} \\ &= q^{-n_0} \frac{2K}{\pi} \sqrt{m_{A,r} m_{B,r}}. \text{ qed} \end{aligned}$$

Application 2.

The equation

$$k(Ax^2 + By^2 + Cx + Dy + E) + l = n \quad (17)$$

has $r = r_{A,B} \left(\frac{n-l}{k} + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right)$ solutions.

In general if $P_N(x) = \sum_{k=0}^N a_k x^k$ is a polynomial with integer coefficients and there exists exactly one integer n' such that $P_N(n') = n$ then

$$P_N(Ax^2 + By^2 + Cx + Dy + E) = n \quad (18)$$

has

$$r_{A,B} \left(n' + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right) \quad (19)$$

integer solutions, (including 0).

Furthermore, if the equation $P_N(n') = n$ has integer solutions $n' = n'_1, n'_2, \dots, n'_s$ with $s \leq N$, then the number of representations of n by (18) will be

$$r = \sum_{i=1}^s r_{A,B} \left(n'_i + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right). \quad (20)$$

Any non-integer solution n' to $P_N(n') = n$ leads to no representation (18) and hence makes no contribution to the sum (20).

Consider now the function $\sum_{n=0}^{\infty} q^{n^\nu}$, $\nu \in \mathbf{N}$ and $\nu > 2$. Then

$$\left(\sum_{n=0}^{\infty} q^{n^\nu} \right)^2 = \sum_{t=0}^{\infty} \left(\sum_{a^\nu + b^\nu = t} 1 \right) q^t. \quad (21)$$

Set

$$\mathbf{1}_\nu(t) := T \left(\sum_{a^\nu + b^\nu = t} 1 \right). \quad (22)$$

Then $P_\nu(n) := \mathbf{1}_\nu(n) = 1$, when n is of the form m^ν , (m positive integer) and 0 otherwise. This leads to

Theorem 4.

The number of representations of n by $x^\nu + y^\nu$, where x, y are non-negative integers, is

$$r_\nu(n) = \sum_{k=0}^n P_\nu(k) P_\nu(n-k). \quad (23)$$

Proof.

From (22) we get

$$\sum_{a^\nu + b^\nu = n} 1 = T^{(-1)}(\mathbf{1}_\nu(n)) \quad (24)$$

where $T^{(-1)}(f_n)$ is the n -th Taylor coefficient of f^2 . Hence from the Leibniz formula

$$T^{(-1)}(f_n) = \sum_{k+l=n} f_k f_l \quad (25)$$

Also, in the case where n is the ν -th power of a positive integer, formula (23) gives $r_\nu(n^\nu) = 0$, (Fermat-Wiles theorem). qed

In general if $A(n)$ is a polynomial with positive integer coefficients then the equation $A(a) + A(b) = n$, where a, b, n are non negative integers, has

$$\sum_{A(a)+A(b)=n} 1 = \sum_{k=0}^n G_A(k)G_A(n-k) \quad (26)$$

solutions. The function $G_A(n)$ is such that $G_A(n) = 1$ if there exists a positive integer m such that $n = A(m)$ and 0 otherwise.

3 Representations by some cubic and quintic forms

In this section we give two formulas similar to Jacobi's (Theorem 2) for the representation of a positive integer by a cubic and by a quintic form. The results of Section 2 can be generalized to higher order terms under certain conditions. Historically, there are some results known regarding the cubic case. For example it is known that the Diophantine equation

$$ax^3 - by^3 = n \quad (27)$$

for a, b, n integers, has a finite number of solutions (see [6]).

From the Fermat-Wiles theorem it is known that

$$x^3 + y^3 = z^3 \quad (28)$$

has only trivial solutions i.e. $\{x, 0, x\}$ and $\{0, x, x\}$.

Also a result of Euler states that the equation

$$x^3 + y^3 = z^2 \quad (29)$$

admits a parametric solution in integers (see [5] p.578-579).

We proceed by stating and proving

Theorem 5.

The number representations of n by the form $x^3 + y^3$ (x, y non negative integers) is

$$r_3(n) = \sum_{\substack{d|n \\ d^3 - 4n = 0}} 1 + 2 \cdot \sum_{\substack{d|n \\ d^3 - 4n \neq 0}} S\left(\frac{-d^2 + 4\frac{n}{d}}{3}\right) \quad (30)$$

where $S(n) = 1$ if n is perfect square and 0 otherwise.

Proof.

One has $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ so if we set $u = x+y$ and $v = x^2 - xy + y^2$, x, y are given by

$$x = \frac{1}{6} \left(3u - \sqrt{-3u^2 + 12v} \right), \quad y = \frac{1}{6} \left(3u + \sqrt{-3u^2 + 12v} \right)$$

Hence we get the necessary and sufficient conditions for u, v to determine an integer $n = uv$ that can be expressed as the sum of two cubes.

The quintic case is the similar the cubic. We have

Theorem 6.

The number of representations of n by the form $x^5 + y^5$ (x, y non-negative integers) is $r_5(0) = 1$ and if n positive integer

$$r_5(n) = - \sum_{\substack{d|n \\ d^5 - 16n = 0}} 1 + \\ + 2 \cdot \sum_{\substack{d|n \\ d^5 - 16n \neq 0}} \mathbf{X}_N \left(\frac{5d - \sqrt{-25d^2 + 10\sqrt{5d^4 + 20\frac{n}{d}}}}{10} \right) S \left(5d^4 + 20\frac{n}{d} \right) \times \\ \times S \left(-25d^2 + 10\sqrt{5d^4 + 20\frac{n}{d}} \right) \quad (31)$$

where \mathbf{X}_N is the characteristic function on the positive integers.

4 Generalized Triangular Numbers

We call

$$t_m(n) = \frac{n^2 + mn}{2}, \quad n = 0, 1, 2, \dots, \text{ with } m = 0, 1, 2, \dots \quad (32)$$

the m -triangular numbers. We interested in the number of representations of a certain non-negative integers n as the sum of N in m -triangular numbers

$$n = \sum_{k=1}^N t_m(x_k) = \sum_{k=1}^N \frac{x_k^2 + mx_k}{2}, \text{ where } x_k \in \mathbf{Z} \text{ and } m = 0, 1, 2, \dots \quad (33)$$

The case of $m = 1, N = 2, 3, 4, \dots$ is the well known representation of n into simple triangular numbers (1-triangular numbers) and has been treated by many mathematicians (see [11]). The case $m = 0$ is Jacobi's N -square theorem. At this point we drop the notation $r_{A,B}(n)$ we used above and denote the number of representations of n in (33) by $r_{m,N}(n)$. Also we denote $r(n)$ of (8) as $r_2(n)$, but the symbol $r_N(n)$ is left as in previous sections i.e is the number of representations of n by the diagonal form $\sum_{k=1}^N x_k^2$. Also, we recall the definition of certain theta functions studied by Ramanujan (see [14] pg.36):

Definition 1.

If $|q| < 1$, then

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}} \quad (34)$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \quad (35)$$

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty} \quad (36)$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n). \quad (37)$$

Consider now the Jacobi triple product formula (see [15] pg.169-172):

$$\sum_{n=-\infty}^{\infty} q^{n^2+zn} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + q^{2n+1-z})(1 + q^{2n+1+z}) \quad (38)$$

where $|q| < 1$.

In case $z = 2p + 1$, with p a non-negative integer we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2+(2p+1)n} &= f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2(n-p)})(1 + q^{2(n+p)+2}) = \\ &= f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2(n-p)})(1 + q^{2(n+p)+2}) = \\ &= f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2n}) \prod_{n=0}^{p-1} (1 + q^{2(n-p)}) \frac{\prod_{n=0}^{\infty} (1 + q^{2n+2})}{\prod_{n=0}^{p-1} (1 + q^{2n+2})} = \\ &= 2q^{-p(p+1)} f(-q^2) (-q^2; q^2)_{\infty}^2. \end{aligned}$$

Since

$$\prod_{n=0}^{p-1} \frac{1 + q^{2(n-p)}}{1 + q^{2n+2}} = q^{-p(p+1)}$$

and

$$(-q^2; q^2)_{\infty} = \frac{(q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}}$$

we have

Proposition 4.

If $|q| < 1$ and $p = 0, 1, 2, \dots$, then

$$\sum_{n=-\infty}^{\infty} q^{t_{2p+1}(n)} = 2q^{-p(p+1)/2} \frac{f(-q^2)^2}{f(-q)} = 2q^{-p(p+1)/2} \psi(q) \quad (39)$$

Proof.

The first equality follows from above discussion. For the second equality we have

$$\frac{f(-q^2)^2}{f(-q)} = \frac{(q^2; q^2)_\infty^2}{(q; q)} = \frac{(q^2; q^2)}{(q; q^2)} = \psi(q),$$

since

$$(q; q^2)_\infty \cdot (-q; q)_\infty = 1 \quad (40)$$

Proposition 5.

If $|q| < 1$ and $p \in \mathbf{Z}$, then

$$\sum_{n=-\infty}^{\infty} q^{t_{2p}(n)} = q^{-p^2/2} \phi(q^{1/2}) \quad (41)$$

Proof.

The proof is elementary since

$$\sum_{n=-\infty}^{\infty} q^{(n+p)^2} = \phi(q)$$

when $|q| < 1$ and $p \in \mathbf{Z}$.

Using the above Propositions we can generalize all the results in [11]. Starting from the $2p+1$ -triangular numbers and Proposition 4, we immediately have

Theorem 7.

If $N \geq 2$ is an integer and $\delta_N(n)$ denotes the the number of representations of n as N 1-triangular numbers

$$n = \sum_{k=1}^N \frac{x_k^2 + x_k}{2} \quad (42)$$

and if $r_{2p+1, N}(n)$ denotes the number of representations of the positive integer n as N $2p+1$ -triangular numbers

$$n = \sum_{k=1}^N \frac{x_k^2 + (2p+1)x_k}{2}, \quad (43)$$

then

$$r_{2p+1, N}(n) = \delta_N \left(n + \frac{Np(p+1)}{2} \right). \quad (44)$$

For example we have

Example 1.

i) The number of ways of representing n in the form

$$n = \frac{x^2 + (2p+1)x}{2} + \frac{y^2 + (2p+1)y}{2} \quad (45)$$

is

$$r_{2p+1,2}(n) = 4d_1(8(n+p^2+p)+2) - 4d_3(8(n+p^2+p)+2) \quad (46)$$

where

$$d_a(n) = \sum_{\substack{d|n \\ d \equiv a(4)}} 1, \text{ with } a = 1, 3 \quad (47)$$

ii)

$$r_{2p+1,4}(n) = \sigma_1(2n + 4p(p+1) + 1) \quad (48)$$

where $\sigma_\nu(n) = \sum_{d|n} d^\nu$ is the divisor function.

Continuing from Section 2 we get expressions for $r_{2p,N}(n)$.
Since we know

$$r_N(n) = T \left(\sum_{k_1+k_2+\dots+k_N=n} r(k_1)r(k_2)\dots r(k_N) \right), \quad (49)$$

we obtain from Proposition 5

Theorem 8.

It is known that $r_N(2n)$ is the number of ways to represent the positive integer n as the sum of N 0-triangular numbers

$$n = \sum_{k=1}^N \frac{x_k^2}{2}. \quad (50)$$

The number of representations of n by

$$n = \sum_{k=1}^N \frac{x_k^2 + 2px_k}{2} \quad (51)$$

is

$$r_{2p,N}(n) = r_N(2n + Np^2). \quad (52)$$

Continuing in this way, from Jacobi's two-square theorem we know that

$$r_2(n) = \sum_{d-\text{odd}, d|n} (-1)^{\frac{d-1}{2}},$$

if $n = 1, 2, \dots$ and $r_2(0) = 1$. Combining the above results we obtain next

Theorem 9.

The number of representations $s_m(n)$ of n in the form

$$\frac{x^2 + mx}{2} + \frac{y^2 + my}{2} \quad (53)$$

is

i) If m is even

$$s_m(n) = 4 \sum_{\substack{d \mid \left(n + \frac{m^2}{4}\right) \\ d \equiv 1(2)}} (-1)^{\frac{d-1}{2}} \quad (54)$$

ii) If m is odd

$$s_m(n) = 4 \sum_{\substack{d \mid (m^2 + 4n) \\ d \equiv 1(4)}} 1 - 4 \sum_{\substack{d \mid (m^2 + 4n) \\ d \equiv 3(4)}} 1 \quad (55)$$

Theorem 10.

The number of representations of n as a sum of four m -triangular numbers

$$\frac{x^2 + mx}{2} + \frac{y^2 + my}{2} + \frac{z^2 + mz}{2} + \frac{w^2 + mw}{2} \quad (56)$$

is

i) If $m = 2p$, $p = 0, 1, 2, \dots$,

$$r_{2p,4}(n) = r_4(2n + 4p^2), \quad (57)$$

where (see [12]):

$$r_4(n) = 8 \sum_{d \mid n} d, \text{ if } n \text{ is odd} \quad (58)$$

and

$$r_4(n) = 24 \sum_{\substack{d \mid n \\ d \equiv 1(2)}} d, \text{ if } n \text{ is even} \quad (59)$$

ii) If $m = 2p + 1$, $p = 0, 1, 2, \dots$,

$$r_{2p+1,4}(n) = \sigma_1(2n + 4p(p + 1) + 1) \quad (60)$$

From Theorem 10 we get the next

Theorem 11.

For any given integer m , each non negative integer n can be represented as the sum

of four m -triangular numbers.

Note. Theorem 11 is a generalization of the Lagrange's famous four square theorem.

Theorem 12.

The number of representation of n in the form

$$n = \frac{x^2 + 2px}{2} + \frac{y^2 + 2py}{2} + \frac{z^2 + 2pz}{2} \quad (61)$$

is

$$r_{2p,3}(n) = r_3(2n + 3p^2) \quad (62)$$

where

$$r_3(n) = \begin{cases} 24h(-n), & n \equiv 3(8) \\ 12h(-4n), & n \equiv 1, 2, 5, 6(8) \\ 0, & n \equiv 7(8) \end{cases} \quad (63)$$

and where $h(n)$ is the class number of n .

5 An Exponential Method

Proposition 6.

In general, if $q = e^{-2x}$, $x > 0$ and $X(n)$ is arithmetic function then

$$\sum_{n=1}^{\infty} X(n) \frac{n^2}{\sinh^2(nx)} = -\frac{d^2}{dx^2} \log \left(\prod_{n=1}^{\infty} (1 - e^{-2nx})^{X(n)} \right) \quad (64)$$

Proof.

See [18]. qed

Let

$$\chi_0(n) = \begin{cases} -2 & \text{if } n \equiv 1(mod 4) \\ 3 & \text{if } n \equiv 2(mod 4) \\ -2 & \text{if } n \equiv 3(mod 4) \\ 1 & \text{if } n \equiv 0(mod 4). \end{cases} \quad (65)$$

Then if $q = e^{-2x}$ we get (see [17],[18])

$$\sum_{n=1}^{\infty} \chi_0(n) \frac{n^2}{\sinh^2(nx)} = -\frac{d^2}{dx^2} \log \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right) = -\frac{d^2}{dx^2} \log (\theta_3(e^{-2x})) \quad (66)$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sinh^2(nx)} = -\frac{d^2}{dx^2} \log \left(\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} \right) \quad (67)$$

A relationship between the hyperbolic sine function series and theta functions is:

$$\text{If } \chi_{k,h}(n) := \begin{cases} 1, & \text{if } n \equiv 0, k+h, k-h \pmod{2k} \\ 0, & \text{otherwise,} \end{cases} \quad (68)$$

then

$$\sum_{n=1}^{\infty} \frac{\chi_{k,h}(n)n^2}{\sinh^2(nx)} = -\frac{d^2}{dx^2} \log \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+hn} \right) \quad (69)$$

when $k > h$, $k \in \mathbf{N}$, $h \in \mathbf{Z}$.

Assume that not both k, h are even or odd, then from [18]

$$\log \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+hn} \right) = -\sum_{n=1}^{\infty} f_{k,h}(n) q^n \quad (70)$$

where

$$f_{k,h}(n) := \frac{1}{n} \sum_{d|n} \chi_{k,h}(d) d. \quad (71)$$

Hence

$$\sum_{n=-\infty}^{\infty} q^{kn^2+hn} = \exp \left(-\sum_{n=1}^{\infty} (-1)^n f_{k,h}(n) q^n \right). \quad (72)$$

If we define $T_0(a_n)$ to be such that

$$\exp \left(-\sum_{n=1}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} T_0(a_n) x^n, \quad (73)$$

then

Theorem 13.

The number of representations of $n \in \mathbf{N}$ in the form

$$n = \sum_{l=1}^N k_l x_l^2 + h_l x_l \quad (74)$$

with $k_l > |h_l| > 0$, k_l, h_l not both even or both odd, $\forall l = 1, 2, \dots, N$ is

$$r(n) = T_0 \left(\sum_{l=1}^N (-1)^l f_{k_l, h_l}(n) \right) \quad (75)$$

where $f_{k,h}$ is that of (71) and $\chi_{k,h}$ is that of (68).

Examples.

i) For example, the number of representations of n in the form

$$10x^2 + 11y^2 + x + 4y \quad (76)$$

is

$$r(n) = T_0 \left(\frac{(-1)^n}{n} \sum_{d|n} \chi_{10,1}(d)d + \frac{(-1)^n}{n} \sum_{d|n} \chi_{11,4}(d)d \right) \quad (77)$$

ii) Another example is the number of representations of n by the form

$$3x^2 - 2x + 3y^2 - 2y \quad (78)$$

which is

$$r(n) = T_0 (2(-1)^n \sigma_6^*(n) + 2(-1)^n I_6(n)), \quad (79)$$

with

$$\sigma_a^*(n) = \frac{1}{n} \sum_{d|n} \left(\frac{d}{a^2} \right) d \quad (80)$$

where $I_a(ka) = 1$ for $k = 1, 2, \dots$ and is otherwise 0.

iii) For

$$n = 2x^2 - x + 2y^2 - y \quad (81)$$

we have

$$r(n) = T_0 (2(-1)^n \sigma_2^*(n) + 2(-1)^n I_4(n)) \quad (82)$$

iv) For

$$n = 4x^2 + 3y^2 + 3x + 2y \quad (83)$$

we have

$$r(n) = T_0 \left(\frac{(-1)^n}{n} \sum_{d|n} \chi_{4,3}(d)d + \frac{(-1)^n}{n} \sum_{d|n} \chi_{3,2}(d)d \right). \quad (84)$$

6 Asymptotic Expansion of $\sum_{n \leq x} r_2(n)$

In this section we provide asymptotic formulas relating to the mean value of $r_2(n)$, using a formula of Hardy (see [16]).

$$\sum_{n \leq x} r_2(n) = \pi x + x^{1/2} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) \quad (85)$$

If $a, b \in \mathbf{R}$, then we define

$$M_s(a, b) = \sum_{k \text{ odd}, k=1}^{\infty} (-1)^{\frac{k+1}{2}} \frac{\cos(a + b\sqrt{k})}{k^s} \quad (86)$$

$$N_s(a, b) = \sum_{k \text{ odd}, k=1}^{\infty} (-1)^{\frac{k+1}{2}} \frac{\sin(a + b\sqrt{k})}{k^s} \quad (87)$$

and

$$P_s(a, b) = \sum_{n=1}^{\infty} \frac{M_s(a, b\sqrt{n})}{n^s}, \quad Q_s(a, b) = \sum_{n=1}^{\infty} \frac{N_s(a, b\sqrt{n})}{n^s}. \quad (88)$$

Next we prove

Theorem 14.

$$\begin{aligned} R(x) = \sum_{n \leq x} r_2(n) - x\pi &= \frac{x^{1/4}}{\pi} P_{3/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right) + \sum_{s=1}^N \frac{(-1)^s c_1(2s) P_{s+3/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right)}{2^{4s} \pi^{2s+1} x^{s-1/4}} - \\ &- \sum_{s=0}^N \frac{(-1)^s c_1(2s+1) Q_{s+5/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right)}{2^{4s+2} \pi^{2s+2} x^{s+1/4}} + O\left(c_1(2N) 4^{-N} x^{-N-1/2}\right) \end{aligned} \quad (89)$$

where $c_1(m) = (-1)^m \frac{(-\frac{1}{2})_m (\frac{3}{2})_m}{m!}$.

Proof.

From (85) and (9) we have

$$\begin{aligned} \sqrt{x} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) &= \sqrt{x} \sum_{n=1}^{\infty} \left(\sum_{d \text{ odd}, d|n} (-1)^{\frac{d-1}{2}} \right) \frac{1}{\sqrt{n}} J_1(2\pi\sqrt{nx}) = \\ &= \sqrt{x} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{n(2m-1)}} J_1\left(2\pi\sqrt{n(2m-1)x}\right) = \\ &= \sqrt{x} \sum_{p \text{ odd}, p=1}^{\infty} \frac{(-1)^{\frac{p-1}{2}}}{\sqrt{np}} J_1(2\pi\sqrt{np}x) \end{aligned} \quad (90)$$

The function $J_1(x)$ has the following asymptotic expansion as $x \rightarrow \infty$

$$\begin{aligned} J_1(x) &= \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{3\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n)}{(2x)^{2n}} - \right. \\ &\quad \left. - \sin\left(x - \frac{3\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n+1)}{(2x)^{2n+1}} \right] \end{aligned} \quad (91)$$

The error due to stopping the summation at any term is the order of magnitude of that term multiplied by $1/x$. Hence, using (91) in (90) we get (89).

Setting $N = 1$ in (89), we get

$$R(x) = - \sum_{\substack{n, p=1 \\ p \text{ odd}}}^{\infty} \left[\frac{105(-1)^{\frac{p+1}{2}} \sin\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{4096\pi^3 (np)^{9/4} x^{5/4}} - \frac{15(-1)^{\frac{p+1}{2}} \cos\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{256\pi^2 (np)^{7/4} x^{3/4}} \right] +$$

$$+ \frac{3(-1)^{\frac{p+1}{2}} \sin\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{8\pi(np)^{5/4}\sqrt[4]{x}} - \frac{2(-1)^{\frac{p+1}{2}}\sqrt[4]{x} \cos\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{(np)^{3/4}}] + O\left(x^{-3/4}\right)$$

where $p = 2l + 1$. Therefore, we have

Proposition 7.

The Gauss circle problem reduces to finding the rate of convergence of $R(x) = \frac{1}{x^{1/4}} \left(\sum_{n \leq x} r_2(n) - \pi x \right)$, which is equivalent to that of

$$S(x) = \sum_{n=1}^{\infty} \frac{r_2(n) \cos\left(2\pi\sqrt{n}x + \frac{\pi}{4}\right)}{n^{3/4}} = \sum_{n,l=1}^{\infty} \frac{(-1)^{l-1} \cos\left(2\pi\sqrt{n(2l-1)}x + \frac{\pi}{4}\right)}{(n(2l-1))^{3/4}}. \quad (92)$$

We can write

$$S(x) = \sum_{n,l=1}^{\infty} \frac{1}{n^{3/4}} \left(\frac{\cos\left(2\pi\sqrt{n(4l+1)}x + \frac{\pi}{4}\right)}{(4l+1)^{3/4}} - \frac{\cos\left(2\pi\sqrt{n(4l-1)}x + \frac{\pi}{4}\right)}{(4l-1)^{3/4}} \right)$$

Also if we set

$$\theta(n, l, x) = \cos\left(2\pi\sqrt{n(4l-1)}x + \frac{\pi}{4}\right) - \cos\left(2\pi\sqrt{n(4l+1)}x + \frac{\pi}{4}\right) \quad (93)$$

then, because

$$\lim_{l \rightarrow \infty} (4l+1)^{11/4+1} \left(\frac{1}{(4l-1)^{3/4}} - \frac{1}{(4l+1)^{3/4}} - \frac{3/2}{(4l+1)^{3/4+1}} \right) = \frac{21}{8},$$

we can write

$$\frac{1}{(4l-1)^{3/4}} - \frac{1}{(4l+1)^{3/4}} = \frac{3/2}{(4l+1)^{3/4+1}} + O\left(\frac{1}{(4l+1)^{11/4}}\right). \quad (94)$$

Hence we get

$$S(x) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{\cos\left(2\pi\sqrt{n(4l+1)}x + \frac{\pi}{4}\right)}{n^{3/4}} \left[\frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} \right] + \\ + \sum_{n,l=1}^{\infty} \frac{\theta(n, l, x)}{n^{3/4}(4l+1)^{3/4}}.$$

But, if we set $\sigma'_\nu(n) = \sum_{d \equiv 1(4), d|n} d^\nu$ then (since the sequences we use involving the big O -symbol are bounded above; also see [9] pg.135-136):

$$\sum_{n,l=1}^{\infty} \frac{\cos\left(2\pi\sqrt{n(4l+1)}x + \frac{\pi}{4}\right)}{n^{3/4}} \left[\frac{1}{(4l-1)^{3/4}} - \frac{1}{(4l+1)^{3/4}} \right] =$$

$$\begin{aligned}
& \frac{3}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{n^{3/4}(4l+1)^{3/4+1}} + O \left(\sum_{l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{n^{3/4}(4l+1)^{11/4}} \right) \right] = \\
& \frac{3}{2} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{n \cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{n^{3/4+1}(4l+1)^{3/4+1}} + \sum_{n=1}^{\infty} O \left(\sum_{l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{n^{3/4}(4l+1)^{11/4}} \right) = \\
& \frac{3}{2} \sum_{n=1}^{\infty} \frac{\sigma'_1(n) \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{3/4+1}} + O \left(\sum_{n=1}^{\infty} \frac{\sigma'_2(n) \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{11/4}} \right) = \\
& \frac{3}{2} \sum_{n=1}^{\infty} \frac{\sigma'_1(n) \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{1+\delta} n^{3/4-\delta}} + O \left(\sum_{n=1}^{\infty} \frac{\sigma'_2(n) \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{2+\delta} n^{3/4-\delta}} \right) \quad (95)
\end{aligned}$$

Now if we keep in mind the inequality $\sigma'_a(n) = o(n^{a+\delta})$ and assume that the sums

$$D_M(x) := \sum_{n=1}^M \frac{\cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{3/4-\delta}} \quad (96)$$

are uniformly bounded, when $M \rightarrow \infty$, for every $\delta > 0$ sufficiently small, then using Abel's test (see [13] pg.346) the two series in (95) are uniformly convergent. Also

$$\begin{aligned}
& \left| \sum_{n,l=1}^{\infty} \frac{\theta(n,l,x)}{n^{3/4}(4l+1)^{3/4}} \right| \leq \\
& \left| \sum_{n,l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{(n(4l+1))^{3/4}} \right| + C \left| \sum_{n,l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l-1)x} + \frac{\pi}{4} \right)}{(n(4l-1))^{3/4}} \right| = \\
& = O \left(\sum_{n,l=1}^{\infty} \frac{\cos \left(2\pi \sqrt{n(4l+1)x} + \frac{\pi}{4} \right)}{(n(4l+1))^{3/4}} \right) \\
& = O \left(\sum_{n=1}^{\infty} \frac{\sigma'_0(n) \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{\delta} n^{3/4-\delta}} \right) = O \left(\sum_{n=1}^{\infty} \frac{1}{n^{\delta-\epsilon}} \frac{\cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{3/4-\delta}} \right) \quad (97)
\end{aligned}$$

Since it is known that exists C_0 such that $\sigma'_0(n) \leq C_0 n^{\epsilon}$, for $0 < \epsilon < \delta$, if the sum (96) is uniformly bounded, again using Abel's test we get the uniform convergence of (97).

The Euler-Maclaurin formula for a function F having 4 continuous derivatives in the interval (a, m) states

$$\sum_{k=a}^M F(k) = \int_a^{M+1} F(t) dt + \frac{1}{2} (F(M+a) + F(a)) +$$

$$+ \frac{1}{12}(F'(M+a) - F'(a)) - \frac{1}{120} \sum_{k=0}^{M-1} F^{(4)}(a+k+\xi) \quad (98)$$

with $0 < \xi < 1$.

If we set

$$F(t) = \frac{\cos(2\pi\sqrt{tx} + \frac{\pi}{4})}{\sqrt{t}}$$

then

$$\int_a^{M+1} \frac{\cos(2\pi\sqrt{(t-1)x} + \frac{\pi}{4})}{\sqrt{t-1}} dt = \frac{\sin(2\pi\sqrt{xM} + \frac{\pi}{4})}{\pi\sqrt{x}} - \frac{\sin(2\pi\sqrt{x} + \frac{\pi}{4})}{\pi\sqrt{x}}.$$

Also

$$F'(t) = -\frac{\cos(2\pi\sqrt{xt} + \frac{\pi}{4})}{2t^{3/2}} - \sqrt{x}\pi \frac{\sin(2\pi\sqrt{xt} + \frac{\pi}{4})}{t}$$

and

$$\begin{aligned} F^{(4)}(t) = & \frac{\pi^4 x^2 \cos(\frac{1}{4}\pi(8\sqrt{tx} + 1))}{t^{5/2}} - \frac{5\pi^3 x \sqrt{tx} \sin(\frac{1}{4}\pi(8\sqrt{tx} + 1))}{t^{7/2}} + \\ & + \frac{105\pi\sqrt{tx} \sin(\frac{1}{4}\pi(8\sqrt{tx} + 1))}{8t^{9/2}} - \frac{45\pi^2 x \cos(\frac{1}{4}\pi(8\sqrt{tx} + 1))}{4t^{7/2}} + \\ & + \frac{105 \cos(\frac{1}{4}\pi(8\sqrt{tx} + 1))}{16t^{9/2}}. \end{aligned}$$

Hence the Euler-Maclaurin summation formula assures us that the $\lim_{M \rightarrow \infty} D_M(x)$ exists i.e. the series $D(x)$ converges. Note that we don't use $n^{3/4}$ in the sum $D_M(x)$, but \sqrt{n} . One can see that this doesn't change anything (we proceed with \sqrt{n} instead of $n^{3/4}$ for avoid showing large formulas).

From all the arguments in the present paragraph we are able to prove

Proposition 8.

For $\delta > 0$ small enough the series

$$D(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}} \quad (99)$$

is convergent.

Further if exist always fixed $\delta_0 > 0$ small enough, such for every $0 < \delta < \delta_0$ the partial sums

$$D_M(x) = \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}} \quad (100)$$

are uniformly bounded, then for every function $f(x)$ such that $\lim_{x \rightarrow \infty} f(x) = +\infty$ holds

$$\sum_{n \leq x} r_2(n) = x\pi + O(x^{1/4}f(x)), \text{ as } x \rightarrow \infty \quad (101)$$

Proof.

If $D_M(x)$ is uniformly bounded then it is also uniformly convergent (because of parameter δ) and we have

$$\lim_{x \rightarrow \infty} \frac{D(x)}{f(x)} = \lim_{x \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta} f(x)} = \sum_{n=1}^{\infty} \lim_{x \rightarrow \infty} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta} f(x)} = 0.$$

An example of such function $f(x)$ is $\log_n(x) = \underbrace{\log(\log(\dots(\log(x))))}_{n\text{-times}}$, for fixed large n .

Lemma 1. (see [9] pg.145)

Suppose that $\lambda_1, \lambda_2, \dots$ is a nondecreasing sequence of real numbers with limit infinity, that c_1, c_2, \dots is an arbitrary sequence of real or complex numbers, and that $f(x)$ has a continuous derivative for $x \geq \lambda_1$. Put

$$C(x) = \sum_{\lambda_n \leq x} c_n \quad (102)$$

where the summation is over all n for which $\lambda_n \leq x$. Then for $x \geq \lambda_1$,

$$\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x)f(x) - \int_{\lambda_1}^x C(t)f'(t)dt \quad (103)$$

Theorem 15.

For every $\delta > 0$ sufficiently small the sum (100) is always bounded (uniformly bounded) in M and x . By this the asymptotic formula (101) is true and the Gauss circle problem is solved.

Proof.

Let the function

$$f(t) = \frac{\cos(2\pi t\sqrt{a} + \frac{\pi}{4})}{t^{3/2}} \quad (104)$$

It's first derivative is

$$f'(t) = -\frac{2\pi\sqrt{a} \sin(2\pi\sqrt{a}t + \frac{\pi}{4})}{t^{3/2}} - \frac{3 \cos(2\pi\sqrt{a}t + \frac{\pi}{4})}{2t^{5/2}} \quad (105)$$

From Lemma setting $y = \sqrt{M}$ we have

$$\sum_{\sqrt{n} \leq y} 1 = y^2 = M \quad (106)$$

Also with $y = \sqrt{M}$ we have

$$\sum_{\sqrt{n} \leq y} f(\sqrt{n}) = \left(\sum_{\sqrt{n} \leq x} 1 \right) f(x) - \int_1^x t^2 f'(t) dt =$$

$$= \sum_{n \leq M} f(\sqrt{n}) = Mf(\sqrt{M}) - \int_1^{\sqrt{M}} t^2 f'(t) dt$$

Hence

$$\begin{aligned} \sum_{n \leq M} f(\sqrt{n}) &= \sum_{n \leq M} \frac{\cos(2\pi\sqrt{na} + \frac{\pi}{4})}{n^{3/4}} = Mf(\sqrt{M}) - \int_1^{\sqrt{M}} t^2 f'(t) dt = \\ &= \frac{1}{\sqrt{2}\sqrt[4]{a}} \left(-2F_C(2\sqrt[4]{a}) + 2F_C(2\sqrt[4]{aM}) + 2F_S(2\sqrt[4]{a}) - 2F_S(2\sqrt[4]{aM}) \right) + \\ &\quad + \frac{1}{\sqrt{2}} (\cos(2\pi\sqrt{a}) - \sin(2\pi\sqrt{a})) \end{aligned} \quad (107)$$

where $F_C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$ and $F_S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$ are the Fresnel- C, S functions.

But function (107) is absolutely bounded when $M = 1, 2, \dots$ and $a > 0$ by some universal constant (we mean $\sqrt{2}$). Hence for the sum

$$G(h, x, M) := \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-h}} \quad (108)$$

it holds that $G(0, x, M)$ is bounded in $M \in \mathbf{N}$ and $x > 0$. Using the mean-value theorem there exists $0 < \xi < \delta$ such that

$$\left| \frac{G(\delta, x, M) - G(0, x, M)}{\delta} \right| = |\partial_h G(\xi, x, M)|$$

Hence

$$\begin{aligned} |G(\delta, x, M)| &\leq \delta |\partial_h G(\xi, x, M)| + |G(0, x, M)| = \\ &= \delta \left| \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{1/2}} \frac{\log(n)}{n^{1/4-\xi}} \right| + |G(0, x, M)| \end{aligned} \quad (109)$$

If we consider next the function

$$f_1(t) = \frac{\cos(2\pi t\sqrt{a} + \frac{\pi}{4})}{t} \quad (110)$$

we can show as above that

$$\sum_{n \leq M} f_1(\sqrt{n}) \quad (111)$$

is uniformly bounded and (109) becomes

$$|G(\delta, x, M)| \leq \delta \left| \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{1/2}} \frac{\log(n)}{n^{1/4-\xi}} \right| + |G(0, x, M)| =$$

$$= O \left(\delta \left| \sum_{n=1}^M \frac{\cos \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right)}{n^{1/2}} \right| \right) + |G(0, x, M)| < \infty \quad (112)$$

uniformly in M and x , when $0 < \delta < \frac{1}{4}$.

Hence $D_M(x)$ are bounded in M and x when δ is sufficiently small, and the theorem is proved.

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